One-ended subforests and whatnot

Part 4

Let G be a loc. fin. aperiodic (:= every component is infinite) pmp graph on a standard probability space (X, μ).

(1) Matt presented. If G is hyperfinite and nowhere two-ended, then G admits a Borel a.e. one-ended spanning subforest $F \subseteq G$ with $E_G = E_F$.

(2) Jenna presented. If G has superquadratic growth ($\exists c > 0 \; \forall x \in X \; \forall r \in \mathbb{N} \; |B_r^G(x)| \geq c r^2$), then G admits a Borel a.e. one-ended subforest.

These two cases cover all loc. fin. pmp graphs that are nowhere two ended; in other words, hyperfinite $\cup$ super-quadratic = all. Why?

(3) Kaimanovich – Elek. Not μ-hyperfinite $\iff$ $\exists$ pos.-measured Borel B s.t. $\varphi(G|_B) > 0$.

(4) Jenna presented. $\varphi(G) > 0 \xRightarrow{\text{local-global bridge (pmp)}}$ G has exponential growth, in fact $\forall^\mu x \in X \; \forall r, \; |B_r^G(x)| \geq (1 + \varphi(G))^r$.

Recall. The isoperimetric constant of G is
$$\varphi(G) := \inf_{A \subseteq X} \frac{\mu(\partial A)}{\mu(A)},$$
where $\partial A := \{x \in X \setminus A : x \, G \, A\}$ and A ranges over positively measured Borel finite-component sets for G, i.e. $G|_A$ is component-finite.

Main theorem (about pmp one-ended subforest). Let G be a loc. fin. aperiodic pmp graph. If G is nowhere 2-ended, then G admits a Borel a.e. 1-ended spanning subforest. And the converse is also true.

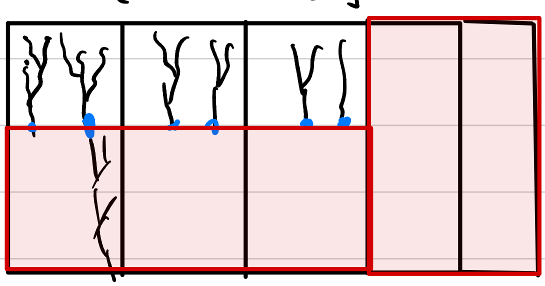
Proof. By Zorn's lemma (= measure exhaustion), we get a maximal collection $\mathcal{B}$ of pairwise disjoint pos.-measured Borel sets B s.t. $\varphi(G|_B) > 0$. $\mathcal{B}$ is ctbl because $\mu(X) < \infty$, hence $\bar{B} := [\bigcup \mathcal{B}]_{E_G}$ is Borel. $A := X \setminus \bar{B}$ is invariant & $G|_A$ is hyperfinite by (3)$\Rightarrow$.

By Matt's talk, $G|_A$ admits a Borel a.e. one-ended sp. subforest. By Jenna's: $\forall B \in \mathcal{B}$, $G|_B$ admits hence $G|_{\cup \mathcal{B}}$ admits it too (just take the union over all $B \in \mathcal{B}$).

Claim. If G admits a one-end sp. subforest on a set $C \subseteq X$ then G admits it on $[C]_{E_G}$.

Proof. For each $x \in [C] \setminus C$ take lex-least path (using a ctbl Borel edge-coloring, by Feldman–Moore) to the set C. These paths form a forest F' on $[C] \setminus C$. Let F be a one-ended sp. subforest on C. Then we show that $F' \cup F$ is still a one-ended subforest. Enough to show that each connected component of F' is finite.

$[C]$

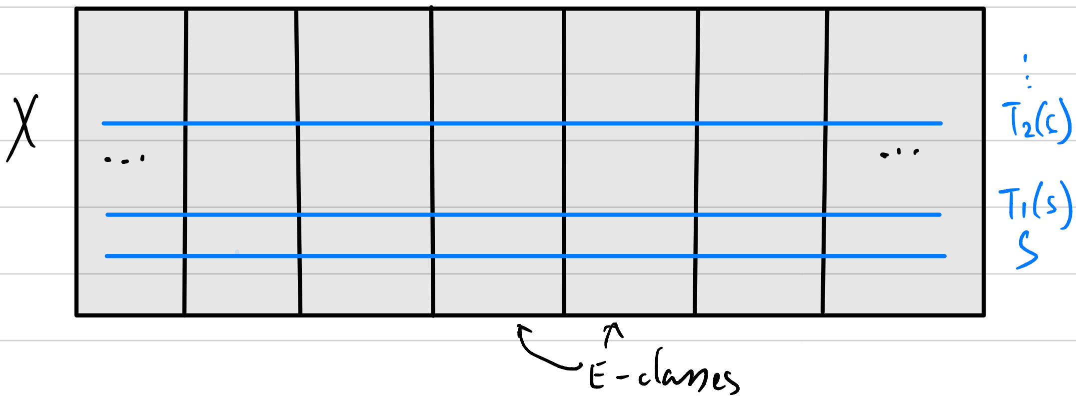


C

But $E_{F'}$ is smooth and smooth pmp equivalence relations are finite.

Lemma. If E is a smooth ctbl Borel eq. rel. that pmp then it's finite.

Proof. Let $S \subseteq X$ be a transversal for E, i.e. a Borel set that meets each E-class at exactly one point. Suppose that some E-classes are infinite; restricting to this set,



We may assume that every E-class is infinite. Then, the Luzin-Novikov uniformization theorem (equivalently, Feldman-Moore) gives Borel injections $T_n : S \hookrightarrow X$ with $\mathrm{graph}(T_n) \subseteq E$ such that $T_0 = \mathrm{id}_S$ and $X = \bigsqcup_{n \in \mathbb{N}} T_n(S)$ — disjoint union.
Then

$$1 \underset{\text{p.}}{=} \mu(X) = \sum_{n \in \mathbb{N}} \mu(T_n(S)) \underset{\text{m.p.}}{=} \sum_{n \in \mathbb{N}} \mu(S) = \infty \cdot \mu(S),$$

a contradiction. $\square$